International Journal of Theoretical Physics, Vol. 34, No. 7, 1995

Anyonic Variables and the Quantum Hyperplane

E. Ahmed,[1] A. S. Hegazi,[1] and S. S. El-Mahdy[1]

Received February 2, 1995

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1. INTRODUCTION

The idea of grading (Van Oystean and Nastassecu, 1982) is well known in algebra. There, a Z_n grading of a ring R is a collection of subrings R_i such that $R = \Sigma R_i$ and $R_i R_j \subseteq R_{i+j}$.

An analogous idea can be followed for variables. Commuting variables (real or complex) correspond to Z_1 grading, while anticommuting variables (Taylor and Ferrara, 1982) correspond to Z_2 grading. Recently we have defined semionic variables (Ahmed *et al.*, 1993), which correspond to Z_4 grading. In this note we generalize our previous results to variables with Z_n grading. These variables are called anyonic variables. These variables are introduced here and shown to form a representation of the quantum hyperplane.

2. ANYONIC VARIABLES

The variables $\theta_1, \theta_2, \ldots$ are said to be π/n anyonic variables if

$$\theta_k \theta_l = \exp\left[i \frac{\pi}{n} S(k - l) \right] \theta_l \theta_k \tag{2.1}$$

[1] Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 35516, Egypt.

0020-7748/95/0700-0977$07.50/0

where

$$S(k-l) = \begin{cases} 1, & k > l \\ -1, & k < l \\ 0, & k = l \end{cases} \tag{2.2}$$

It is straightforward to see that

$$S(k-l) + S(l-k) = 0 \tag{2.3}$$

Hence the definition (2.1) is consistent. Bosonic (commuting) variables correspond to taking the limit $n \to \infty$, while fermionic (anticommuting) variables correspond to $n = 1$.

Differentiation and integration of anyonic variables are defined as follows:

$$\frac{\partial 1}{\partial \theta_l} = 0, \qquad \frac{\partial \theta_k}{\partial \theta_l} = \delta_k^l, \qquad \frac{\partial (\theta_k)^2}{\partial \theta_l} = (1 + e^{i(\pi/n)S(k-l)})\theta_k \delta_k^l \tag{2.4}$$

With these definitions, it follows that

$$\frac{\partial (\theta_l)^p}{\partial \theta_j} = \frac{1 - e^{i\pi p/n}}{1 - e^{i\pi/n}} \delta_l^j (\theta_l)^{p-1} \tag{2.5}$$

Notice that when $p = 2n$ the right-hand-side of this last equation identically vanishes. Hence we impose, for any π/n anyonic variables, the following condition:

$$(\theta_j)^{2n} = 0 \tag{2.6}$$

For anticommuting variables, $n = 1$ and we regain the familiar result $(\theta_j)^2 = 0$.

Translation invariance and equation (2.6) suggest the following definition for integration over anyonic variables:

$$\int (\theta_j)^{2n-1} \, d\theta_l = \delta_{jl} \tag{2.7}$$

and the integration of any other power of θ_j is zero. For $n = 1$ the familiar Brezin integral (Taylor and Ferrara, 1982) is regained.

3. THE QUANTUM HYPERPLANE

The quantum hyperplane is defined in Manin (1989) and Faddeev *et al.* (1988) as the set of coordinates x_l, $l = 1, 2, \ldots$, such that

$$x_l x_j = q x_j x_l, \qquad l < j \tag{3.1}$$

The corresponding differential dx_i satisfies

$$dx_i dx_j = -\frac{1}{q} dx_j dx_i, \qquad i < j \tag{3.2}$$

On the other hand, the noncommutative differential calculus advocated in Wess and Zumino (1990) and Zumino (1991) states that in general the coordinates x_i obey the commutation relation

$$r_{pj} \equiv x_p x_j - B_{pj}^{kl} x_k x_l = 0 \tag{3.3}$$

for some tensor B_{ij}^{kl}. These commutation relations lead to the consistency condition

$$\partial_m r_{ij} = 0 \tag{3.4}$$

Furthermore, the differentials dx_i in general satisfy

$$x_p\, dx_j = C_{pj}^{kl}\, dx_k\, x_l \tag{3.5}$$

A straightforward comparison shows that for anyonic variables of type π/n we have

$$q = \exp\left[i\frac{\pi}{n} S(l - j) \right]$$

$$B_{pj}^{kl} = \exp\left[i\frac{\pi}{n} S(p - j) \right] \delta_j^k \delta_p^l \tag{3.6}$$

The consistency condition (3.4) is satisfied for anyonic variables due to the property (2.3). The tensor C_{jl}^{pk} is given by

$$C_{jl}^{pl} = \delta_l^p \delta_j^k \exp\left[i\frac{\pi}{n} S(p - j) \right] \tag{3.7}$$

The R-matrix for the quantum group $GL_q(n)$ is (Manin, 1989; Faddeev *et al.*, 1988; Wess and Zumino, 1990; Zumino, 1991)

$$R_{kl}^{ij} = \delta_k^j \delta_l^i [1 + (q - 1)\delta^{ij}] + \left(q - \frac{1}{q} \right) \delta_k^j \delta_l^i \Theta(j - i) \tag{3.8}$$

where

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{3.9}$$

The matrix R satisfies the Yang–Baxter relation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{3.10}$$

where the tensor product notation has been used.

Thus we have shown that anyonic variables form a representation of the quantum hyperplane. An interesting correspondence between particles and variables is as follows: Commuting variables correspond to bosons. Anticommuting variables correspond to fermions. Anyonic variables correspond to particles with fractional states of the type known in the fractional Hall effect (Laughlin, 1988) and superconductivity (Fradkin, 1991).

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